Probability & Statistics (1)

Random Variables (I)

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Random Variables

- 很多時候，我們不是很在乎中間的過程發生甚麼事，而是更重視結果是甚麼。例如: 你在乎的是骰子點數和為7的事件，而非今天到底兩個骰子實際上的數字是多少，像是{1,6}, {2,5}, {3,4}, {4,3}, {5,2}, {6,1}。或是一枚硬幣出現多少次正面等。

[from Textbook]

These quantities of interest, or, more formally, these real-valued functions defined on the sample space, are known as *random variables*.

Random Variables

- **範例一**

假設今天投擲三枚硬幣，令事件Y為人頭出現的次數，則Y為一隨機變數，當Y等於0, 1 ,2 ,3所相對應的機率。

Solution:

$$P\{Y=0\}=P\{(T,T,T)\}=\frac{1}{8}$$

$$P\{Y=1\}=P\{(H,T,T),(H,H,T),(T,T,H)\}=\frac{3}{8}$$

$$P\{Y=2\}=P\{(H,T,H),(T,H,T),(T,T,H)\}=\frac{3}{8}$$

$$P\{Y=3\}=P\{(H,H,H)\}=\frac{1}{8}$$

$$1=P\left(\bigcup_{i=0}^{3}\{Y=i\}\right)=\sum_{i=0}^{3}P\{Y=i\}$$

Random Variables

- **範例二**

一個摸彩箱裡面裝有編號1到20號的彩球，現在要從裡面拿出三顆球，取出不放回，至少一顆的編號要大於等於17號，試問其機率為何?

Solution:

令X為編號最大的球被選到且為一隨機變數，其值域為$1-20$之間。

$$P\{X=i\}=\frac{\binom{i-1}{2}}{\binom{20}{3}}, where\ i=3,\dots,20$$

Random Variables

$$P\{X=20\}=\frac{\binom{20-1}{2}}{\binom{20}{3}}=\frac{\binom{19}{2}}{\binom{20}{3}}=\frac{3}{20}\approx 0.150$$

$$P\{X=19\}=\frac{\binom{19-1}{2}}{\binom{20}{3}}=\frac{\binom{18}{2}}{\binom{20}{3}}=\frac{51}{380}\approx 0.134$$

$$P\{X=18\}=\frac{\binom{18-1}{2}}{\binom{20}{3}}=\frac{\binom{17}{2}}{\binom{20}{3}}=\frac{34}{285}\approx 0.119$$

$$P\{X=17\}=\frac{\binom{17-1}{2}}{\binom{20}{3}}=\frac{\binom{16}{2}}{\binom{20}{3}}=\frac{2}{19}\approx 0.105$$

$$P\{X\geq 17\}\approx 0.150+0.134+0.119+0.105=0.508$$

Random Variables

- **範例三**

我們用投擲硬幣進行一連串的獨立試驗，假設出現一次人頭或投擲n次，出現人頭的機率為p，令X為硬幣投擲的次數，則X為一隨機變數且建立在$1,2,3,\ldots,n$其中之一，試問X出現的機率為何?

Solution:

$$P\{X=1\}=P\{H\}=p$$
$$P\{X=2\}=P\{(T,H)\}=(1-p)p$$
$$P\{X=3\}=P\{(T,T,H)\}=(1-p)^2p$$
$$P\{X=n-1\}=P\{(\underbrace{T,\ldots,T}_{n-2},H)\}=(1-p)^{n-2}p$$
$$P\{X=n\}=P\{(\underbrace{T,\ldots,T}_{n-1},T),(\underbrace{T,\ldots,T}_{n-1},H)\}=(1-p)^{n-1}$$

Random Variables

$$P\left(\bigcup_{i=1}^{n}\{X=i\}\right)=\sum_{i=1}^{n}P\{X=i\}$$

$$=\sum_{i=1}^{n-1}p(1-p)^{i-1}+(1-p)^{n-1}$$

$$=p\left[\frac{1-(1-p)^{n-1}}{1-(1-p)}\right]+(1-p)^{n-1}$$

$$=1-(1-p)^{n-1}+(1-p)^{n-1}$$

$$=1$$

Random Variables

- **範例四**

某百貨公司舉辦周年慶出了N種不同的優惠券，每一次只能拿一張且每次拿到優惠券與前次拿到優惠卷為獨立事件，也就是說每次拿到的優惠券會在這N種之一。令隨機變數T為要收集一定的數量優惠全才能將每一種優惠券至少拿一張。但我們可以不直接計算$P\{T = n\}$，而我們可以考慮T會大於n的時候，事件$A_1, A_2, \dots, A_N$: A_j為在前n種優惠券中沒有收集到j種優惠券的事件，$j = 1,2, \dots, N$。

Random Variables

Solution:

$$P\{T > n\} = P\left(\bigcup_{i=1}^{N} A_j\right)$$

$$= \sum_{j} P(A_j) - \sum_{j_1<j_2} \sum P(A_{j1}A_{j2}) + \cdots$$

$$+ (-1)^{k+1} \sum_{j_1<j_2<\cdots<j_k} \sum P(A_{j1}A_{j2} \dots A_{jk}) \dots + (-1)^{N+1} P(A_1 A_2 \dots A_N)$$

Random Variables

$$since\ P(A_j) = \left(\frac{N-1}{N}\right)^n, P(A_{j1}A_{j2}) = \left(\frac{N-2}{N}\right)^n, P(A_{j1}A_{j2} \ldots A_{jk}) = \left(\frac{N-k}{N}\right)^n$$

$$P\{T > n\}$$

$$= N\left(\frac{N-1}{N}\right)^n - \binom{N}{2}\left(\frac{N-2}{N}\right)^n + \binom{N}{3}\left(\frac{N-3}{N}\right)^n - \cdots + (-1)^N \binom{N}{N-1}\left(\frac{N-1}{N}\right)^n$$

$$= \sum_{i=1}^{N-1} \binom{N}{i}\left(\frac{i}{N}\right)^n (-1)^{i+1}$$

$$then, P\{T > n-1\} = P\{T = n\} + P\{T > n\} \Rightarrow P\{T = n\} = P\{T > n-1\} - P\{T > n\}$$

Random Variables

令D_n為一隨機變數，代表在收集一定數量優惠卷中包含前n種優惠券；因此，我們先只聚焦在k種優惠卷被收集到$P\{D_n = k\}$，這個集合會包含前n種不同優惠券被收集到，則:

$$\begin{cases} A: each\ is\ one\ of\ these\ k\ types \\ B: each\ of\ these\ k\ types\ is\ represented \end{cases}$$

$$P(A) = \left(\frac{k}{N}\right)^n, P(B|A) = 1 - \sum_{i=1}^{k-1} \binom{N}{k} \left(\frac{k-i}{k}\right) (-1)^{i+1}$$

$$P\{D_n = k\} = \binom{N}{k} P(AB) = \binom{N}{k} \left(\frac{k}{N}\right)^n \left[1 - \sum_{i=1}^{k-1} \binom{N}{k} \left(\frac{k-i}{k}\right) (-1)^{i+1}\right]$$

Discrete Random Variables

A random variable that can take on at most a countable number of possible values is said to be discrete.

For a discrete variable X, we define the ***probability mass function* (機率質量函數)** $p(a)$ of X by

$$p(a) = P\{X = a\}$$

Discrete Random Variables

The probability mass function $p(a)$ is positive for at most a countable number of values of a. That is, if X must assume one of the values $x_1, x_2, \ldots$, then

$$p(x_i) \geq 0 \; for \; i = 1,2, \ldots$$
$$p(a) = 0 \; for \; all \; other \; values \; of \; x$$

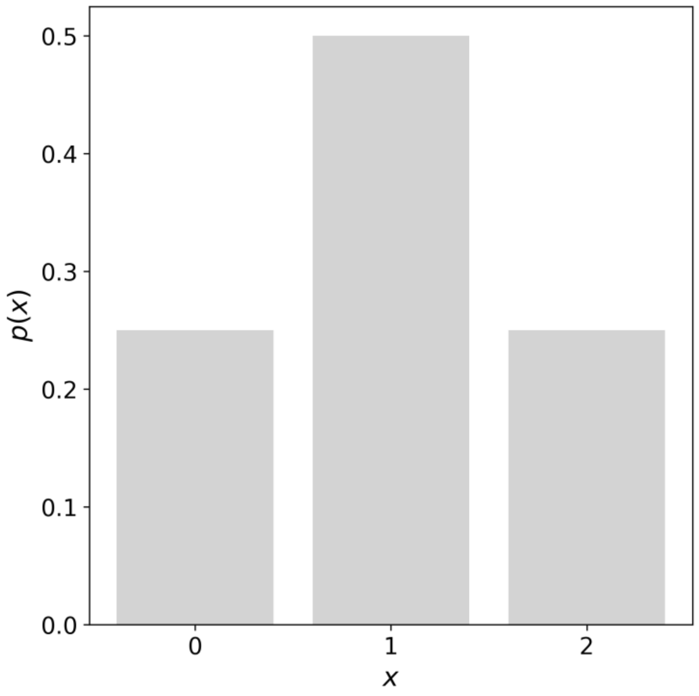
Since X must take on one of the values x_i, we have

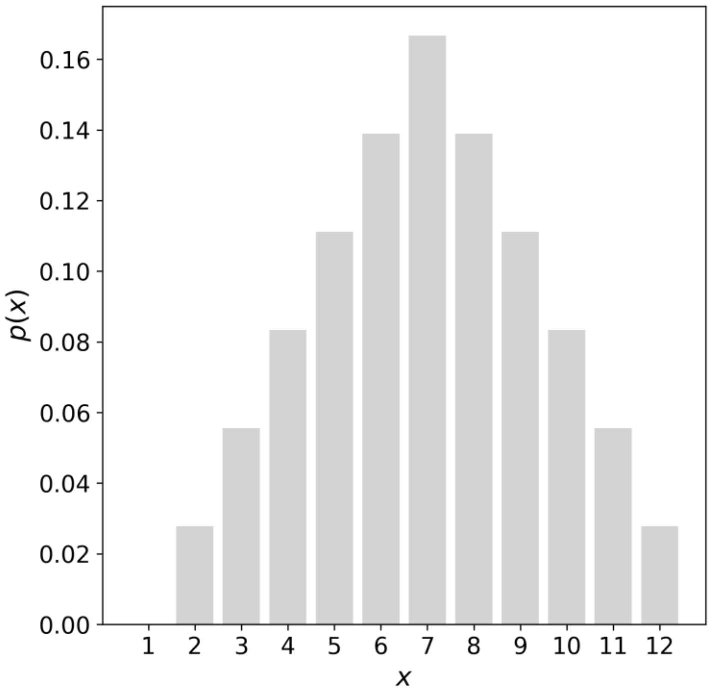
$$\sum_{i=1}^{\infty} p(x_i) = 1$$

Discrete Random Variables

我們一般都是使用長條圖來表現機率質量函數的分布狀況，例如:

$$p(0) = \frac{1}{4}, p(1) = \frac{1}{2}, p(2) = \frac{1}{4}$$





Discrete Random Variables

- 範例五

令random variable X的probability mass function為

$$p(i) = c\frac{\lambda^i}{i!}, where\ i = 0,1,2,\ldots, and\ \lambda > 0$$

試求出$(a)\ P\{X = 0\}\ and\ (b)\ P\{X > 2\}$

Solution:

$$since \sum_{i=0}^{\infty} p(i) = 1, then\ c\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} = 1$$

Discrete Random Variables

$$c\sum_{i=0}^{\infty}\frac{\lambda^i}{i!}=1$$

$$Since\ e^x=\sum_{i=0}^{\infty}\frac{x^i}{i!},implies\ that\ ce^{\lambda}=1\Rightarrow c=e^{-\lambda}$$

$$(a)P\{X=0\}=e^{-\lambda}\frac{\lambda^0}{0!}=e^{-\lambda}$$

$$(b)P\{X>2\}=1-P\{X\le 2\}=1-P\{X=0\}-P\{X=1\}-P\{X=2\}$$

$$=1-e^{-\lambda}-\lambda e^{-\lambda}-\frac{\lambda^2e^{-\lambda}}{2}$$

Discrete Random Variables

The ***cumulative distribution function* (累積分布函數)** of F 可以用 $p(a)$表示為

$$F(a) = \sum_{all\ x \le a} p(x)$$

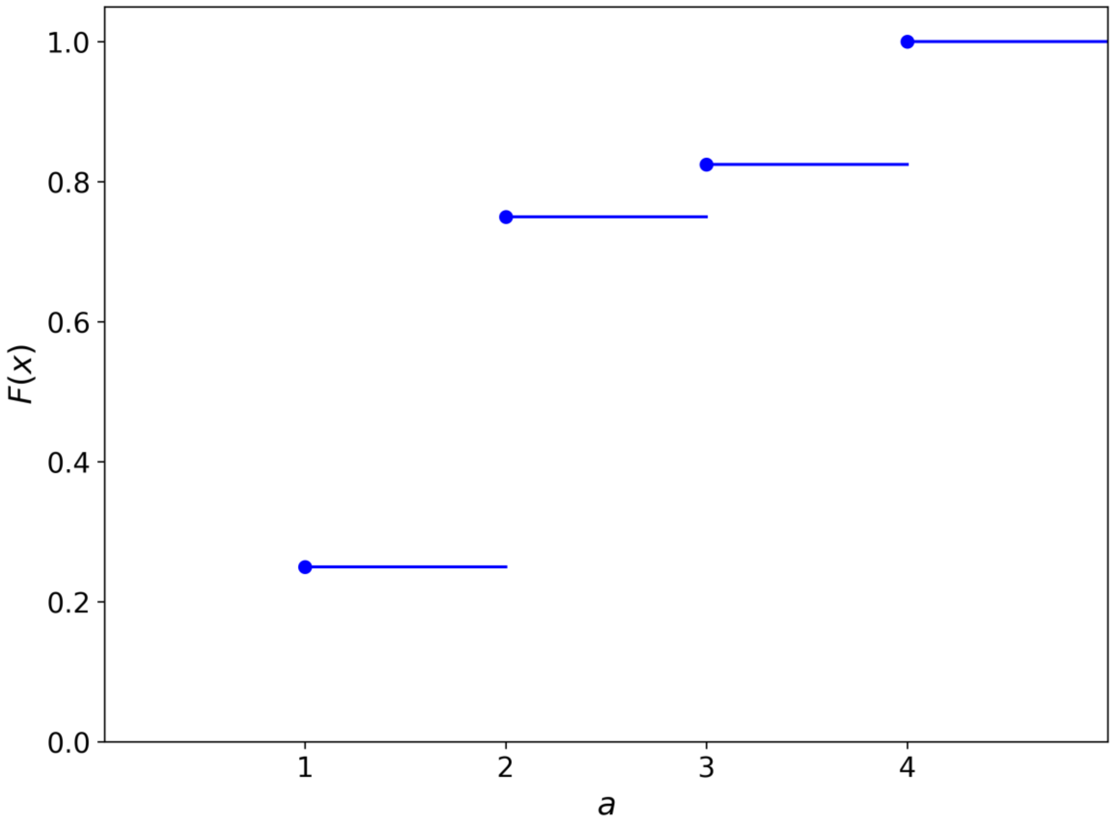
If X is a discrete random variable whose possible values are $x_1, x_2, x_3, \ldots$, where $x_1 < x_2 < x_3 < \cdots$, then the distribution function F of X is a step function. That is, the value of F is constant in the intervals $[x_{i-1}, x_i)$ and then take a step of size $p(x_i)$ at x_i.

Discrete Random Variables

令X的PMF為$p(1)=\frac{1}{4}, p(2)=\frac{1}{2}, p(3)=\frac{1}{8}, p(4)=\frac{1}{8}$

則其CDF為

$$F(a)=\begin{cases} 0, a<1 \\ \frac{1}{4}, 1\le a<2 \\ \frac{3}{4}, 2\le a<3 \\ \frac{7}{8}, 3\le a<4 \\ 1, 4\le a \end{cases}$$



Expected Value

***Expected Value* (期望值)**是在機率論中扮演一個非常重要的地位。

If X is a discrete random variable having a probability mass function $p(x)$, then the expectation, or the expected value, of X, denoted by $E[X]$, is defined by

$$E[X] = \sum_{x:p(x)>0} xp(x)$$

換句話說，期望值就可以被視為X的加權平均數。

例如: $p(0) = \frac{1}{2} = p(1), E[X] = 0\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) = \frac{1}{2}$

Expected Value

如果我們兩個事件發生的機率不一樣的時候。。。

$$p(0) = \frac{1}{3};\ p(1) = \frac{2}{3}$$

$$E[X] = 0\left(\frac{1}{3}\right) + 1\left(\frac{2}{3}\right) = \frac{2}{3}$$

所以我們可以定義隨機變數X一定是$x_1, x_2, \dots, x_n$其中一個數值，且每個數值相對應的機率為$p(x_1), p(x_2), \dots, p(x_n)$。可以將$X$想成我們贏一場遊戲的機會。因此我們平均贏每一場遊戲的結果可被表示成:

$$\sum_{i=1}^{n} x_i p(x_i) = E[X]$$

Expected Value

- **範例六**

計算投擲一個骰子的點數期望值$E[X]$。

Solution:

$$Since\ p(1)=p(2)=p(3)=p(4)=p(5)=p(6)=\frac{1}{6}$$

$$E[X]=1\left(\frac{1}{6}\right)+2\left(\frac{1}{6}\right)+3\left(\frac{1}{6}\right)+4\left(\frac{1}{6}\right)+5\left(\frac{1}{6}\right)+6\left(\frac{1}{6}\right)=\frac{7}{2}$$

Expected Value

- **範例七**

定義一個指標變數I為事件A

$$I = \begin{cases} 1 \; if \; A \; occurs \\ 0 \; if \; A^c \; occurs \end{cases}$$

計算$E[I]$

Solution:

$$Since \; p(1) = P(A), p(0) = 1 - P(A)$$
$$E[I] = P(A)$$

Expected Value

- **範例八**

某電視台舉辦兩題知識競賽: 分別為問題1與問題2。參賽者可以選擇任一題目優先回答，再回答另一題。假設參賽者先回答第i題再回答第j題，且$i \neq j$。如果參賽者答對第i題可以得到V_i元，若參賽者再答對第j題可以得到$V_i + V_j$元。假設參賽者答對第i題的機率為$P_i, i = 1,2$，那麼參賽者應該要先回答哪個問題才能極大化獎金?

Expected Value

Solution:

假設參賽者先回答第一個問題，則他贏的機率與獎金為:

$0 \quad \text{with probability } 1 - P_1$

$V_1 \quad \text{with probability } P_1(1 - P_2)$

$V_1 + V_2 \quad \text{with probability } P_1 P_2$

故參賽者贏的獎金期望值為

$V_1 P_1(1 - P_2) + (V_1 + V_2) P_1 P_2$

如果先回答第二個問題

$V_2 P_2(1 - P_1) + (V_1 + V_2) P_1 P_2$

Expected Value

如果先回答第一題比較好的情況

$$V_1P_1(1-P_2)+(V_1+V_2)P_1P_2 \geq V_2P_2(1-P_1)+(V_1+V_2)P_1P_2$$

$$V_1P_1(1-P_2) \geq V_2P_2(1-P_1)$$

$$\frac{V_1P_1}{(1-P_1)} \geq \frac{V_2P_2}{(1-P_2)}$$

假設答對第一題的機率為60%且贏得$200；答對第二題的機率為40%且贏得$100。

$$400=\frac{(100)(0.8)}{0.2}>\frac{(200)(0.6)}{0.4}=300$$

Expected Value

- **範例九**

今天有120人要去校外教學分三台車: 36人搭第一台車；40人搭第二台車；44人搭第三台車。當遊覽車來的時候，學生可以隨機選擇想搭的車，令X為隨機搭某一台車的學生數，求$E[X]$。

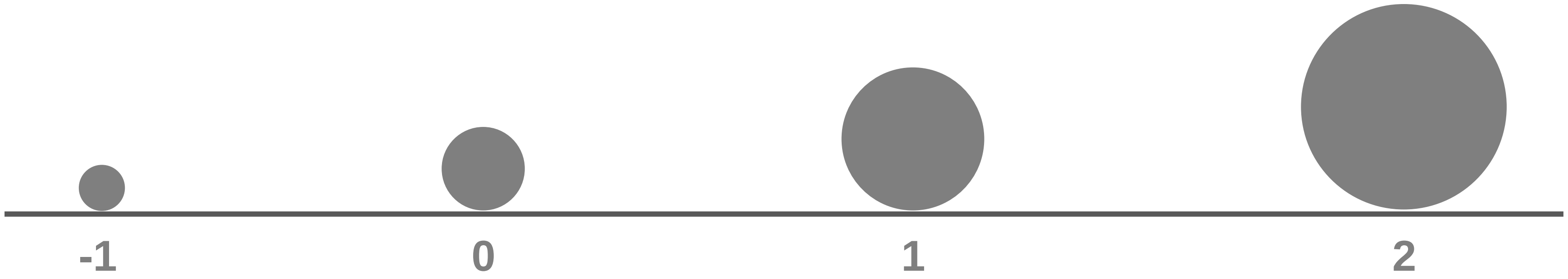
Solution:

$$P\{X=36\}=\frac{36}{120};P\{X=40\}=\frac{40}{120};P\{X=44\}=\frac{44}{120}$$

$$E[X]=36\left(\frac{3}{10}\right)+40\left(\frac{1}{3}\right)+44\left(\frac{11}{30}\right)=\frac{1208}{30}=40.2667$$

Expected Value

- **回想一個問題:**
- 期望值跟物理中一個概念十分相似，你覺得是________。



$p(-1) = 0.1$ $p(0) = 0.25$ $p(1) = 0.30$ $p(2) = 0.35$

$$E[X] = 0.9$$

Expectation of a Function of Random Variable

- 期望值也可以被視為隨機變數的函數。
- 假設隨機變數X可以從$g(X)$產生出來，所以當我們知道$g(X)$的PMF，我們就可以計算出來其期望值$E[g(X)]$。

- **範例十**

令隨機變數X為$-1,0,1$其中之一，其相對應的機率為:

$$P\{X = -1\} = 0.2; P\{X = 0\} = 0.5; P\{X = 1\} = 0.3$$

計算$E[X^2]$。

Expectation of a Function of Random Variable

Solution:

令$Y = X^2$，則隨機變數Y的PMF為

$$P\{Y = 1\} = P\{X = -1\} + P\{X = 1\} = 0.5$$
$$P\{Y = 0\} = P\{X = 0\} = 0.5$$

Hence,

$$E[X^2] = E[Y] = 1(0.5) + 0(0.5) = 0.5$$

Note that,

$$0.5 = E[X^2] \neq (E[X])^2 = 0.01$$

Expectation of a Function of Random Variable

- **Proposition 1**

If X is a discrete random variable that takes on one of the values $x_i, i \geq 1$, with respective probabilities $p(x_i)$, then, for any real-valued function g,

$$E[g(X)] = \sum_i g(x_i)p(x_i)$$

所以前面的範例就可以寫成:

$$E\{X^2\} = (-1)^2(0.2) + 0^2(0.5) + 1^2(0.3)$$
$$= 1(0.2 + 0.3) + 0(0.5) = 0.5$$

Expectation of a Function of Random Variable

- **Proof of Proposition 1**

假設$y_j, j \geq 1$代表$g(x_i)$的不同數值且$i \geq 1$

$$\sum_i g(x_i)p(x_i) = \sum_j \sum_{i:g(x_i)=y_j} g(x_i)p(x_i)$$

$$= \sum_j \sum_{i:g(x_i)=y_j} y_j p(x_i) = \sum_j y_j \sum_{i:g(x_i)=y_j} p(x_i)$$

$$Since\ P(g(X) = y_j)$$

$$= \sum_j y_j P\{g(X) = y_j\} = E[g(X)]$$

Expectation of a Function of Random Variable

- **範例十一**

假設你今天是一家公司的存貨管理的經理，已知如果將每一季每單位產品賣完可得淨利b元；同時，每單位產品沒賣完會淨損ℓ元。令每一季單一通路所產品訂購總量的PMF為$p(i), i \geq 0$，由於當前物料短缺，所以你必須提前存貨，試問你應該要訂購多少產品，以極大化利潤。

Solution:

令X為訂購產品的數量，存貨量為s，則其利潤為$P(s)$

$$P(s) = bX - (s - X)\ell;\ if\ X \leq s$$
$$P(s) = sb;\ if\ X > s$$

Expectation of a Function of Random Variable

期望獲利為

$$E[P(s)] = \sum_{i=0}^{s} [bi - (s - X)\ell]p(i) + \sum_{i=s+1}^{\infty} sbp(i)$$

$$E[P(s)] = (b + \ell)\sum_{i=0}^{s} ip(i) - s\ell\sum_{i=0}^{s} p(i) + sb[1 - \sum_{i=0}^{s} p(i)]$$

$$E[P(s)] = (b + \ell)\sum_{i=0}^{s} ip(i) - (b + \ell)s\sum_{i=0}^{s} p(i) + sb$$

$$E[P(s)] = sb + (b + \ell)\sum_{i=0}^{s}(i - s)p(i)$$

Expectation of a Function of Random Variable

Therefore, …
如果我們存量每加一單位的產品，獲利有甚麼變化?

$$E[P(s+1)] = b(s+1) + (b+\ell)\sum_{i=0}^{s+1}(i-s-1)p(i)$$

$$Since\ i = s+1, then\ (s+1-s-1)p(i) = 0$$

$$= b(s+1) + (b+\ell)\sum_{i=0}^{s}(i-s-1)p(i)$$

So,

$$E[P(s+1)] - E[P(s)] = b - (b+\ell)\sum_{i=0}^{s}p(i) > 0$$

$$where\ stocking\ s+1\ will\ be\ better\ than\ stock\ s.$$

Expectation of a Function of Random Variable

$$E[P(s+1)] - E[P(s)] = b - (b+\ell)\sum_{i=0}^{s} p(i) > 0$$

$where\ stocking\ s+1\ will\ be\ better\ than\ stock\ s.$

$\because sb + (b+\ell)\sum_{i=0}^{s}(i - S)p(i) > 0$

$$\therefore \sum_{i=0}^{s} p(i) < \frac{b}{b+\ell}$$

Expectation of a Function of Random Variable

- **Corollary 1**

If a and b are constant, then

$$E[aX + b] = aE[X] + b$$

Proof:

$$E[aX + b] = \sum_{x:p(x)>0} (ax + b)p(x)$$

$$= a \sum_{x:p(x)>0} xp(x) + b \sum_{x:p(x)>0} p(x)$$

$$since \sum_{x:p(x)>0} p(x) = 1$$

$$= aE[X] + b$$

Expectation of a Function of Random Variable

- 隨機變數X的期望值(expected value)$E[X]$通常拿來代表平均值(mean)或是第一階動差(the first moment)。
- 第n階動差就可以被表示成:

$$E[X^n] = \sum_{x:p(x)>0} x^n p(x)$$

Variance

- 期望值(平均值)$E[X]$雖然可以拿來表示一個分布F的平均產生的結果，但是從這個資訊中，我們無法知道資料的分散程度。

$$X1 = \{50, 50, 50, 50, 50\}$$
$$X2 = \{100, 100, 50, 0, 0\}$$

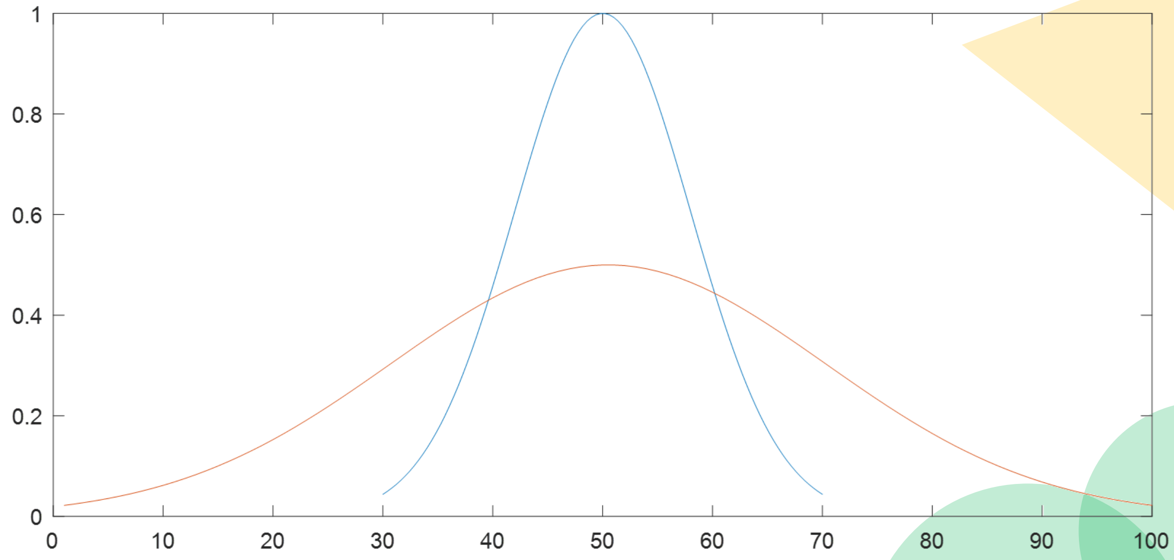
- 所謂資料分散度，我們可以從隨機變數與其期望值之間的差距得知，為了計算平均差異的狀況，我們可以用$E[|X - \mu|], where\ \mu = E[X]$，但是這樣也不是很好計算。因此就有了變異數($variance$)的統計量。

Variance

- **Definition**

If X is a random variable with mean μ, then the variance of X, denoted by $Var(X)$, is defined by,

$$Var(X) = E[(X-\mu)^2]$$



Variance

- An alternative formula for $Var(X)$ is derived as follows:

$$
\begin{aligned}
Var(X) &= E[(X-\mu)^2] \\
&= \sum_x (X-\mu)^2 p(x) \\
&= \sum_x (x^2 - 2\mu x + \mu^2)\, p(x) \\
&= \sum_x x^2 p(x) - 2\mu \sum_x x p(x) + \mu^2 \sum_x p(x) \\
&= E[X^2] - 2\mu^2 + \mu^2 \\
&= E[X^2] - \mu^2 \\
&= E[X^2] - (E[X])^2
\end{aligned}
$$

Variance

- 範例十二

假設X為一個公平骰子投擲出來的點數，試問$Var(X)$。

Solution:

$$E[X] = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = \frac{7}{2}$$

$$E[X^2] = 1^2\left(\frac{1}{6}\right) + 2^2\left(\frac{1}{6}\right) + 3^2\left(\frac{1}{6}\right) + 4^2\left(\frac{1}{6}\right) + 5^2\left(\frac{1}{6}\right) + 6^2\left(\frac{1}{6}\right) = 91\left(\frac{1}{6}\right)$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

Variance

- 變異數的性質

For any constants a and b,

$$Var(aX+b)=a^2Var(X)$$

Proof:

Let $\mu=E[X]$ and note from **Corollary 1** that $E[aX+b]=a\mu+b$.

Therefore,

$$\begin{aligned}Var(aX+b)&=E[(aX+b-a\mu-b)^2]\\&=E[a^2(X-\mu)^2]\\&=a^2E[(X-\mu)^2]\\&=a^2Var(X)\end{aligned}$$

Variance

- 提到變異數，在高中數學課的時候，我們提到一個與變異數很相近的統計量 – **標準差(standard deviation)**，通常會以$SD(X)$來表示X的標準差。

$$SD(X) = \sqrt{Var(X)}$$

The Bernoulli and Binomial Random Variables

- 假設一個試驗可以被區分成「成功(success)」與「失敗(failure)」。令$X = 1$為成功；$X = 0$為失敗，那麼X的PMF就可以被定義為:

$$p(0) = P\{X = 0\} = 1 - p$$

$$p(1) = P\{X = 1\} = p$$

$$where\ p, 0 \leq p \leq 1, is\ the\ probability\ that\ the\ trial\ is\ a\ success$$

滿足上述條件的隨機變數稱為 **Bernoulli random variable**。

$$p \in (0,1)$$

Bernoulli random variable的參數有兩個: $(1, p)$

The Bernoulli and Binomial Random Variables

- 今天是進行n次試驗，每一次成功的機率為p，失敗的機率為$1-p$。如果隨機變數X代表在n次試驗中成功的次數，則X就是binomial random variable。
- **Binomial random variable**的參數有兩個: (n, p)
- Binomial random variable的PMF:

$$p(i) = \binom{n}{i} p^i (1-p)^{n-i}, where\ i = 1,2,3, \dots, n$$

- 根據binomial theorem可以得知其機率總和為1

$$Since\ (x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

$$\sum_{i=0}^{\infty} p(i) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = [p + (1-p)]^n = 1$$

The Bernoulli and Binomial Random Variables

- 範例十三

投擲五枚公平的硬幣。假設所有投擲出來的結果是相互獨立的，試問出現正面的PMF為何?

Solution:

令X為binomial random variable，代表出現正面的次數。

$$P\{X=0\}=\binom{5}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5=\frac{1}{32}$$

$$P\{X=1\}=\binom{5}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4=\frac{5}{32}$$

$$P\{X=2\}=\binom{5}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3=\frac{10}{32}$$

$$P\{X=3\}=\binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2=\frac{10}{32}$$

$$P\{X=4\}=\binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1=\frac{5}{32}$$

$$P\{X=5\}=\binom{5}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0=\frac{1}{32}$$

The Bernoulli and Binomial Random Variables

- **範例十四**

假設你今天在負責生產螺絲的台積電的零件供應商，其良率為99%，且每個螺絲的好壞皆是相互獨立。銷售時是以一包10個做販賣，如果一包2個或2個以上不良品有全額退費的服務，試問有多少的機率會出現退貨?

Solution:

令X為一包中螺絲不良品的數量，符合binomial random variable (10,0.01)，故因不良品而符合退費條件的機率為:

$$1 - P\{X=0\} - P\{X=1\} = 1 - \binom{10}{0}(0.01)^0(0.99)^{10} - \binom{10}{1}(0.01)^1(0.99)^9$$

$$\approx 0.004$$

The Bernoulli and Binomial Random Variables

- 範例十五

假設今天iPhone的鏡頭有n個零組件，每個零組件各自獨立，假設該零組件正常運作的機率為p。整個鏡頭正常運作至少需要一半以上的零組件是有正常功能的。

(a) 試問如果由5個零組件所構成的鏡頭會比由3個零組件構成的鏡頭更加穩定的運作，其p應該為何?

(b) 如何證明由$2k+1$個零組件所構成的鏡頭會比$2k-1$來的更好?

The Bernoulli and Binomial Random Variables

Solution:

(a) 因為計算有多少零組件可以正常運作是一個binomial random variable問題(n, p)

$$\binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p)^1 + p^5$$

$$\binom{3}{2} p^2 (1-p)^1 + p^3$$

The Bernoulli and Binomial Random Variables

If $5 - component\ system$ is better than $3 - component\ one$, then

$$\binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p)^1 + p^5 > \binom{3}{2}p^2(1-p)^1 + p^3$$

$$6p^5 - 15p^4 + 12p^3 - 3p^2 > 0$$

$$3p^2(2p^3 - 5p^2 + 4p - 1) > 0$$

$$3(p-1)^2(2p-1) > 0$$

$$p > \frac{1}{2}$$

The Bernoulli and Binomial Random Variables

(b) 如果需要來證明$2k+1$零組件所構成的鏡頭會比$2k-1$來的更好，那麼我們可以令X為某一零組件數量來讓$2k-1$零組件所構成的鏡頭更正常運作。

$$P_{2k-1}(effective) = P\{X \geq k\} = P\{X = k\} + P\{X \geq k+1\}$$

但是相較於$2k-1$零組件所構成的鏡頭而言，$2k+1$個零組件所構成的鏡頭還有兩個多的零組件尚未被考慮到；因此，

$$P_{2k+1}(effective) = P\{X \geq k+1\} + P\{X = k\}(1-(1-p)^2) + P\{X = k-1\}p^2$$

為了要證明$2k+1-component\ system$比較好

$$P_{2k+1}(effective) - P_{2k-1}(effective) > 0$$

The Bernoulli and Binomial Random Variables

$$P_{2k+1}(effective) - P_{2k-1}(effective) > 0$$

$$\cancel{P\{X \geq k+1\}} + P\{X = k\}(1-(1-p)^2) + P\{X = k-1\}p^2 - (P\{X = k\} + \cancel{P\{X \geq k+1\}}) > 0$$

$$P\{X = k\}(1-(1-p)^2) - P\{X = k\} + P\{X = k-1\}p^2 > 0$$

$$P\{X = k-1\}p^2 - P\{X = k\}(1-p)^2 > 0$$

$$\binom{2k-1}{k-1} p^{k-1}(1-p)^k p^2 - (1-p)^2 \binom{2k-1}{k} p^k (1-p)^{k-1} > 0$$

$$since \binom{2k-1}{k-1} = \binom{2k-1}{k}$$

$$\binom{2k-1}{k} p^k (1-p)^k [p - (1-p)] > 0$$

$$p > \frac{1}{2}$$

Properties of Binomial Random Variables

- 提到Binomial Random Variables的特性，就是要來計算其期望值與變異數。

$$E[X^k] = \sum_{i=0}^{n} i^k \binom{n}{i} p^i (1-p)^{n-i} = \sum_{i=1}^{n} i^k \binom{n}{i} p^i (1-p)^{n-i}$$

$$using\ the\ identity\ i\binom{n}{i} = n\binom{n-1}{i-1}$$

Gives

$$E[X^k] = np \sum_{i=1}^{n} i^{k-1} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i}$$

Properties of Binomial Random Variables

$$E[X^k] = np \sum_{i=1}^{n} i^{k-1} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i}$$

$$Let\ j = i - 1$$

$$= np \sum_{j=0}^{n-1} (j+1)^{k-1} \binom{n-1}{j} p^j (1-p)^{n-1-j}$$

$$let\ Y = \ binomial(n-1, p)$$

$$= npE[(Y+1)^{k-1}]$$

$$\therefore k = 1, E[X] = np$$

Properties of Binomial Random Variables

$since\ E[X^k] = npE[(Y+1)^{k-1}]$

$E[X^2] = npE[(Y+1)^{2-1}]$

$= npE[(Y+1)^1]$

$\because Y = n-1\ and\ E[X] = np$

$= np[(n-1)p+1]$

$\therefore Var(X) = E[X^2] - (E[X])^2$

$= np[(n-1)p+1] - (np)^2$

$= np(np - p + 1 - np) = np(1-p)$

Properties of Binomial Random Variables

- **Proposition 2**

If X is a binomial random variable with parameters (n, p), where $0 < p < 1$, then as k goes from $0\ to\ n$, $P\{X = k\}$ first increases monotonically and then decreases monotonically, reaching its largest value when k is the largest integer less than or equal to $(n + 1)p$.

Properties of Binomial Random Variables

Proof:

為了要證明這個proposition，就要證明$P\{X=k\}/P\{X=k-1\} \geq 1$

$$\frac{P\{X=k\}}{P\{X=k-1\}} = \frac{\frac{n!}{(n-k)!\,k!}p^k(1-p)^{n-k}}{\frac{n!}{(n-k+1)!\,(k-1)!}p^{k-1}(1-p)^{n-k+1}} = \frac{(n-k+1)p}{k(1-p)} \geq 1$$

$$(n-k+1) \geq k(1-p)$$

$$k \leq (n+1)p$$

Computing the Binomial Distribution Function

- 假設X是binomial random variables with (n,p)，其二項式分布函數(binomial distribution function)的CDF:

$$P\{X \le i\} = \sum_{k=0}^{i} \binom{n}{k} p^k (1-p)^{n-k}, where\ i = 0,1,\dots,n$$

$$P\{X = k+1\} = \frac{p}{1-p} \frac{n-k}{k+1} P\{X = k\}$$

Computing the Binomial Distribution Function

$$P\{X=k+1\}=\frac{p}{1-p}\frac{n-k}{k+1}P\{X=k\}$$

- **範例十六**

令X為binomial random variable with $n=6, p=0.4$；

從$P\{X=0\}=(0.6)^6$開始，利用遞迴的方式計算出其PMF。

Solution:

$$P\{X=0\}=(0.6)^6\approx 0.0467$$

$$P\{X=1\}=\frac{4}{6}\frac{6}{1}P\{X=0\}\approx 0.1866$$

$$P\{X=2\}=\frac{4}{6}\frac{5}{2}P\{X=1\}\approx 0.3110$$

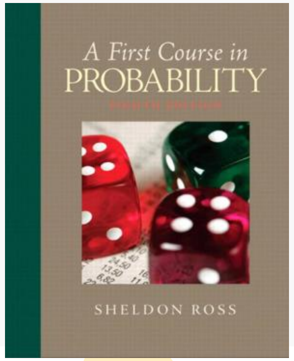
$$P\{X=3\}=\frac{4}{6}\frac{4}{3}P\{X=2\}\approx 0.2765$$

$$P\{X=4\}=\frac{4}{6}\frac{3}{4}P\{X=3\}\approx 0.1382$$

$$P\{X=5\}=\frac{4}{6}\frac{2}{5}P\{X=4\}\approx 0.0369$$

$$P\{X=6\}=\frac{4}{6}\frac{1}{6}P\{X=5\}\approx 0.0041$$

[#8] Assignment



- Selected Problems from Sheldon Ross Textbook [1].

4.13. A salesman has scheduled two appointments to sell encyclopedias. His first appointment will lead to a sale with probability .3, and his second will lead independently to a sale with probability .6. Any sale made is equally likely to be either for the deluxe model, which costs \$1000, or the standard model, which costs \$500. Determine the probability mass function of X, the total dollar value of all sales.

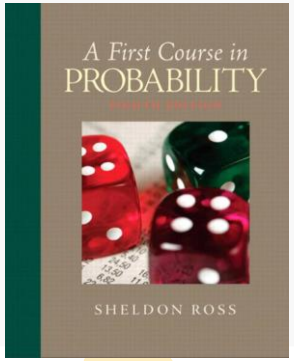
4.17. Suppose that the distribution function of X is given by

$$F(b) = \begin{cases} 0 & b < 0 \\ \dfrac{b}{4} & 0 \le b < 1 \\ \dfrac{1}{2} + \dfrac{b-1}{4} & 1 \le b < 2 \\ \dfrac{11}{12} & 2 \le b < 3 \\ 1 & 3 \le b \end{cases}$$

(a) Find $P\{X = i\}, i = 1, 2, 3$.
(b) Find $P\{\frac{1}{2} < X < \frac{3}{2}\}$.

[1] Sheldon Ross. A First of Course in Probability. 8th edition.

[#8] Assignment



- Selected Problems from Sheldon Ross Textbook [1].

4.19. If the distribution function of X is given by

$$F(b) = \begin{cases} 0 & b < 0 \\ \frac{1}{2} & 0 \le b < 1 \\ \frac{3}{5} & 1 \le b < 2 \\ \frac{4}{5} & 2 \le b < 3 \\ \frac{9}{10} & 3 \le b < 3.5 \\ 1 & b \ge 3.5 \end{cases}$$

calculate the probability mass function of X.

4.28. A sample of 3 items is selected at random from a box containing 20 items of which 4 are defective. Find the expected number of defective items in the sample.

[1] Sheldon Ross. A First of Course in Probability. 8th edition.

Bonus Problem

(1) Using Python programming, to proof “for any constants a and b: $Var(aX + b) = a^2 Var(X)$”

(2) Define a recursive function to compute the values of PMF and CDF of “EX16”. Meanwhile, plot the PMF and CDF with bar chart by using Matplotlib.

Reference

Ross, S. (2010). *A first course in probability*. Pearson.

The End

If you have any questions, please do not hesitate to ask me.

Thank you for your attention))